ON THE DANGER OF COMBINATION RESONANCES

(OB OPASNOSTI KOMBINATSIONNYKH REZONANSOV)

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We have derived the equations for finding characteristic exponents in the case of simple resonance of a linear quasistatic system of second order differential equations with periodic coefficients. For a certain class of equations we investigate the stability on the conjugate resonant frequencies. We consider the effect of friction on the stability of solutions. Following [1], for a wider class of systems we show that the introduction of friction, or an increase of friction already introduced, may result in turning a stable solution into an unstable one. This phenomenon takes place only in combination resonances; it does not occur in the case of simple resonance.

1. Let us consider a system of differential equations

$$\frac{d^2Y}{dt^2} + \mu N(\theta t) \frac{dY}{dt} + (C + \mu P(\theta t)) Y = 0 \qquad (1.1)$$

Here Y is an m-dimensional vector, $\mu \geqslant 0$ and $\theta \geqslant 0$ are real parameters, C is a diagonal matrix and $\omega_s^2 > 0$

$$G = \begin{pmatrix} \omega_1^2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \omega_m^2 \end{pmatrix}, \qquad N(\tau) = \sum_{k=-\infty}^{\infty} N_k e^{ik\tau}$$

$$N(\tau + 2\pi) \equiv N(\tau), \qquad N_k = \| \nu_{js}^{(k)} \|_1^m, \qquad \sum_{k=-\infty}^{\infty} | kN_k | < \infty \qquad (1.2)$$

$$P(\tau + 2\pi) \equiv P(\tau), \qquad P(\tau) = \sum_{k=-\infty}^{\infty} P_k e^{ik\tau}, \qquad P_k = \| \pi_{js}^{(k)} \|_1^m, \qquad \sum_{k=-\infty}^{\infty} | P_k | < \infty$$

The coefficients in (1)1) are real for real values of τ.

Definition 1.1. We shall refer to the system of equations (1.1) and

(1.2) as belonging to class M (i.e. mechanical systems), if its characteristic exponents [2, p.168] are located symmetrically with respect to the imaginary axis.

In Definition 1.1 the condition of symmetry of characteristic exponents may be replaced by the condition of symmetry of multipliers, which are the roots of the characteristic equation [2, p.165], with respect to the unit circle.

System (1.1), describing a reversible mechanical process, will always be of class M. If the Hill's determinant [3], formed for system (1.1), is real for $p = i\omega$ and $\operatorname{Im}\omega = 0$, then system (1.1) belongs to class M.

Let us indicate particular cases of systems of class M.

1) A system of equations (1.1) does not change its form if t is replaced by $-t$. For that it is sufficient to let

$$P(-\tau) \equiv P(\tau), \qquad N(-\tau) \equiv -N(\tau) \tag{1.3}$$

2) System (1.1) is self-conjugate. For that it is sufficient to let

$$P^*(\tau) \equiv P(\tau), \qquad N^*(\tau) \equiv -N(\tau) \equiv \text{const} \tag{1.4}$$

3) An arbitrary system reducible to (1.3) and (1.4) by the transformation

$$X = B(t)Y, \qquad \operatorname{Det} B(t) \neq 0, \qquad B(t + 2\pi\theta^{-1}) \equiv B(t) \tag{1.5}$$

where the matrices $B(t)$ and $B^{-1}(t)$ are bounded, together with their first and second derivatives for $t \in [0, 2\pi\theta^{-1}]$.

The characteristic exponents of system (1.1) are determinate up to an additive term $k\theta i$ ($k = 0, \pm 1, \pm 2, \ldots$). They fall into $2m$ groups which for $\mu = 0$ have the form

$$iw_s + k\theta i, \; -i\omega_s + k\theta i \qquad (s = 1, \ldots, m, \; k = 0, \pm 1, \pm 2, \ldots) \tag{1.6}$$

The characteristic exponents are located in the complex plane symmetrically with respect to the real axis. They change their position continuously with the continuous change of parameters μ and θ. The null solution of system (1.1) of class M cannot be asymptotically stable for $t \to +\infty$. If there is an unbounded solution of the form

$$\exp\{p_1 t\}\varphi_1(t) \qquad (\operatorname{Re} p_1 > 0, \; \varphi(t + 2\pi\theta^{-1}) \equiv \varphi(t)$$

then there necessarily exists a solution of the form

$$\varphi_2(t)\exp\{p_2t\} \quad (\operatorname{Re} p_2 = -\operatorname{Re} p_1 < 0,\ \operatorname{Im} p_2 = \operatorname{Im} p_1,\ \varphi_2(t + 2\pi\theta^{-1}) \equiv \varphi_2(t))$$

When the parameters μ and θ approach the boundary of the region of instability, the characteristic exponents p_1 and p_2 approach the imaginary axis from opposite sides. The exponents coincide when the point of parameters μ and θ reaches the boundary of the instability region. Hence, the equation of the boundary of the instability region of system (1.1) of class M can be obtained from the condition of multiplicity of characteristic exponents. For $\mu = 0$ the condition of coincidence of characteristic exponents has the form

$$n^{-1}|\omega_j \pm \omega_h| = \theta_0 \qquad (j, h = 1, \ldots, m;\ n = 0, 1, 2, \ldots) \tag{1.7}$$

For $\omega_j = \omega_h$ we have the case of simple resonance; if $\omega_j \neq \omega_h$ the resonance is referred to as combination resonance [2, p.341]. The paper concerns itself with the case of resonance when, for a given θ_0, relation (1.7) is fulfilled for a unique set of values of j, h and n and a choice of sign in (1.7).

For a system of equations (1.1) of class M we always have

$$\nu_{ss}^{(0)} = 0 \qquad (s = 1, \ldots, m) \tag{1.8}$$

Definition 1.2. A system of equations (1.1) in which the quadratic form

$$f(y_1, \ldots, y_m) = Y^*N_0Y > 0 \quad \text{for } Y^*Y \neq 0 \tag{1.9}$$

is positive definite shall be referred to as a system of class M with friction, if it belongs to class M for $N_0 \equiv 0$.

If (1.9) is fulfilled we have $\nu_{ss}^{(0)} > 0$ ($s = 1, \ldots, m$). For the characteristic exponent $p_s(\mu, \theta)$, where

$$p_s(0, \theta_0) = \omega_s i, \qquad \omega_s \pm \omega_h \neq k\theta_0 \qquad (h = 1, \ldots, m,\ k = \pm 1, \pm 2, \ldots) \tag{1.10}$$

in the first approximation we obtain the equation

$$p^2 + \mu\nu_{ss}^{(0)}p + \omega_s^2 + \mu\pi_{ss}^{(0)} + O(\mu^2) = 0 \tag{1.11}$$

For $\mu > 0$ we have

$$\operatorname{Re} p_s = \operatorname{Re}\left(-0.5\mu\nu_{ss}^{(0)} + i\sqrt{\omega_s^2 + \mu\pi_{ss}^{(0)} + O(\mu^2)}\right) < 0 \tag{1.12}$$

Therefore, the question of stability of system (1.1) is solved only through the "resonating" exponents p_j which are close to $i\omega j$ for small values of $|\mu|$ and $|\theta - \theta_0|$. The region of instability of a system of class M can be found from the condition Re $p_j > 0$. Let us note that, as a rule, solutions of a system of class M are unstable on the boundary of the region of instability. Solutions of a system of class M with friction are always stable.

Definition 1.3. A frequency θ_0 will be referred to as strongly stable [4], if for matrices $N'(\tau)$ and $P'(\tau)$, which are arbitrary but varied to a sufficiently small extent in (1.1)

$$| N'(\tau) - N(\tau) | < \varepsilon, \qquad | P'(\tau) - P(\tau) | < \varepsilon \qquad (-\infty < t < \infty) \tag{1.13}$$

and which allow system (1.1) to remain in class M, the solutions of system (1.1) will be stable for all values of θ and μ satisfying the condition

$$| \theta - \theta_0 | < \delta, \qquad 0 \leqslant \mu < \delta, \qquad \delta > 0 \tag{1.14}$$

where ε and δ are some positive numbers.

Only a denumerable number of resonant frequencies, namely those of the form (1.7), may not be strongly stable.

Definition 1.4. A frequency θ_0 will be referred to as strongly unstable, if, for matrices $N'(\tau)$ and $P'(\tau)$, which are arbitrary but varied to a sufficiently small extent in (1.1), which satisfy (1.13) and which allow the system (1.1) to remain in class M for any $\varepsilon > 0$ and $\delta > 0$, the values of θ and μ can be found from (1.14) such that the solutions of system (1.1) will be unstable.

In this case a wide region of instability will be adjacent to the resonant frequency θ_0.

Let us agree to call the constant symmetric matrix C a simple positive definite matrix, if a positive definite quadratic form corresponds to it.

2. Below we consider basically the combination resonance with frequencies θ, which are close to the resonant frequency

$$\theta_0 = \theta_{n,j,h} = n^{-1} | \omega_j + \omega_h | \tag{2.1}$$

The formulas for another resonant frequency θ_0^*

$$\theta_0^* = \theta_{n,j,h}^* = n^{-1} | \omega_j - \omega_h, \qquad \omega_j > \omega_h \tag{2.2}$$

can be obtained from the formulas for case (2.1) if ω_h is replaced by $-\omega_h$. The frequencies $\theta_{n,j,h}$ and $\theta^*_{n,j,h}$ will be called conjugate frequencies.

We look for the solution of system (1.1) in the form of a vector series

$$Y(t) = e^{pt} \sum_{s=-\infty}^{\infty} Y_s e^{si\theta t}, \qquad Y_k = \begin{pmatrix} y_{k1} \\ \vdots \\ y_{km} \end{pmatrix} \tag{2.3}$$

Substituting $Y(t)$ from (2.3) into (1.1) and setting the coefficients at different exponents equal to zero, we find an infinite system of linear algebraic equations

$$[E(p + k\theta i)^2 + C] Y_k + \mu \sum_{s=-\infty}^{\infty} [N_{k-s}(p + s\theta i) + P_{k-s}] Y_s = 0$$
$$(k = 0, \pm 1, \pm 2, \ldots) \tag{2.4}$$

Let us change from (2.4) to scalar notation and introduce the designations

$$d_\alpha(k) = [(p + k\theta i)^2 + \omega_\alpha{}^2]^{-1}, \qquad f_{\alpha r}^{k,s} = \nu_{\alpha r}^{(k-s)}(p + s\theta i) + \pi_{\alpha r}^{(k-s)} \tag{2.5}$$

Equations (2.4), solved for $y_{k\alpha}$, take the form

$$y_{k\alpha} = -\mu d_\alpha(k) \sum_{r=1}^{m} \sum_{s=-\infty}^{\infty} f_{\alpha r}^{k,s} y_{sr} \quad (k = 0, \pm 1, \pm 2, \ldots, \ \alpha = 1, 2, \ldots, m) \tag{2.6}$$

We consider system (2.6) in the region

$$|\mu| < \varepsilon_1, \qquad |\theta - \theta_0| < \varepsilon_2, \qquad |p - i\omega_j| < \varepsilon_3 \tag{2.7}$$

It is assumed all the time that, given θ_0 and $\theta_0{}^*$, relations (2.1) and (2.2) are fulfilled only for a unique set of values of j, h, n (j, $h = 1, 2, \ldots, m$; $n = 1, 2, \ldots$). For sufficiently small values of ε_1, ε_2 and ε_3 all the functions $d_\alpha(k)$ (2.5) will be bounded in region (2.7) except two: $d_j(0)$ and $d_h(-n)$, (2.1) and (2.5). In the latter ones the denominators become equal to zero for $\theta = \theta_0$ and $p = i\omega j$. Let us exclude the two equations with the indices $k = 0$, $\alpha = j$ and $k = -n$, $\alpha = h$ from system (2.6). The remaining equations will form a completely regular system [5, p.167] for sufficiently small values of ε_1, ε_2 and ε_3 in (2.7)

$$y_{k\alpha} = -\mu d_\alpha(k) \sum_{r=1}^{m} \sum_{s=-\infty}^{\infty}{}' f_{\alpha r}^{k,s} \cdot y_{sr} - \mu d_\alpha(k) [f_{\alpha j}^{k,0} \cdot y_{0j} + f_{\alpha h}^{k,-n} y_{-nh}] \tag{2.8}$$

Here and below the stroke in the sum means that in the summation the terms with indices $r = j$, $s = 0$ and $r = h$, $s = -n$ are omitted. The solution of system (2.8) can be obtained by the method of successive approximations [5, p. 160]. We have

$$y_{k\alpha} = \Big[-\mu d_\alpha(k) f_{\alpha j}^{k,0} + \mu^2 \sum_{r=1}^{m} \sum_{s=-\infty}^{\infty}{}' d_\alpha(k) f_{\alpha r}^{k,s} d_r(s) f_{rj}^{s,0} - \ldots\Big] y_{0j} +$$
$$+\Big[-\mu d_\alpha(k) f_{\alpha h}^{k,-n} + \mu^2 \sum_{r=1}^{m} \sum_{s=-\infty}^{\infty}{}' d_\alpha(k) f_{\alpha r}^{k,s} d_r(s) f_{rh}^{s,-n} - \ldots\Big] y_{-nh} \qquad (2.9)$$

Substituting (2.9) into the remaining two equations

$$[(p + k\theta i)^2 + \omega_\alpha^2]\, y_{k\alpha} + \mu \sum_{r=1}^{m} \sum_{s=-\infty}^{\infty} f_{\alpha r}^{k,s} y_{sr} = 0 \qquad (2.10)$$

with indices $k = 0$, $\alpha = j$, $k = -n$ and $\alpha = h$ we obtain the system

$$a_{11} y_{0j} + a_{12} y_{-nh} = 0, \qquad a_{21} y_{0j} + a_{22} y_{-nh} = 0 \qquad (2.11)$$

The known quantities a_{sk} $(s, k = 1, 2)$ have the form

$$a_{11} = p^2 + \omega_j^2 + \mu f_{jj}^{(0)} - \mu^2 \sum_{r=1}^{m} \sum_{s=-\infty}^{\infty}{}' f_{jr}^{0,s} d_r(s) f_r^{s,0} + O(\mu^3)$$
$$a_{12} = \mu f_{jh}^{0,-n} - \mu^2 \sum_{r=1}^{m} \sum_{s=-\infty}^{\infty}{}' f_{jr}^{0,s} d_r(s) f_{rh}^{s,-n} + O(\mu^3) \qquad (2.12)$$
$$a_{21} = \mu f_{hj}^{-n,0} - \mu^2 \sum_{r=1}^{m} \sum_{s=-\infty}^{\infty}{}' f_{hr}^{-n,s} d_r(s) f_{rj}^{s,0} + O(\mu^3)$$
$$a_{22} = (p - n\theta i)^2 + w_h^2 + \mu f_{hh}^{-n,-n} - \mu^2 \sum_{r=1}^{m} \sum_{s=-\infty}^{\infty} f_{hr}^{-n,s} d_r(s) f_{rh}^{s,-n} + O(\mu^3)$$

The condition for the existence of the non-zero solution of system (2.11) is

$$a_{11} a_{22} - a_{12} a_{21} = 0 \qquad (2.13)$$

Equation (2.13) will permit the determination of characteristic exponents which are close to $i\omega j$ in region (2.7). The same equation could be obtained from formula (1.8) [6]. The derivation given here is simpler.

3. Here and below we will always assume that for given θ_0 and $\theta_0{}^*$ in (2.1) and (2.2), the values of j, h and n can be chosen in a unique way (except for the interchange of places of ω_j and ω_h). Let us set in (2.13) and (2.12)

$$p = i\omega_j + iz\mu, \qquad \theta = \theta_0 + \lambda\mu, \qquad \theta_0 = n^{-1}(\omega_j + \omega_h) \qquad (3.1)$$

with the accuracy up to small values $O(\mu)$. Equation (2.12) takes the form

$$\begin{vmatrix} i\nu_{jj}^{(0)} + \pi_{jj}^{(0)} / \omega_j - 2z & i\nu_{jh}^{(n)} - \pi_{jh}^{(n)} / \omega_h \\ i\nu_{hj}^{(-n)} + \pi_{hj}^{(-n)} / \omega_j & i\nu_{hh}^{(0)} - \pi_{hh}^{(0)} / \omega_h - 2z + 2\lambda n \end{vmatrix} = 0 \tag{3.2}$$

Equation (3.2) can be obtained from a more general equation (5.8) [3]. For the resonance part of $\theta_0{}^*$ (2.2) the equation for characteristic exponents differs from (3.2) only by the sign at ω_h. Let us introduce the designation

$$g = \left(\frac{\pi_{hj}^{(-n)}}{\omega_j} + i\nu_{hj}^{(-n)}\right)\left(\frac{\pi_{jh}^{(n)}}{\omega_h} - i\nu_{jh}^{(n)}\right) \tag{3.3}$$

The bar denotes the complex conjugate. Quadratic equation (3.2) has the solution

$$z_{1,2} = \frac{1}{4}\left\{\frac{\pi_{jj}^{(0)}}{\omega_j} - \frac{\pi_{hh}^{(0)}}{\omega_h} + i\,(\nu_{jj}^{(0)} + \nu_{hh}^{(0)}) + 2\lambda n \pm \right.$$
$$\left. \pm \left[\left(\frac{\pi_{jj}^{(0)}}{\omega_j} + \frac{\pi_{hh}^{(0)}}{\omega_h} + i\,(\nu_{jj}^{(0)} - \nu_{hh}^{(0)}) - 2\lambda_h\right)^2 - 4g\right]^{1/2}\right\} \tag{3.4}$$

For system (1.1) of class M without friction $\nu_{jj}{}^{(0)} = \nu_{hh}{}^{(0)} = 0$. Since z_1 and z_2 must be located symmetrically with respect to the real axis, the expression under the radical sign in (3.4) must be real for $\mathrm{Im}\,\lambda = 0$, i.e. $\mathrm{Im}\,g = 0$ (3.3). Hence we have the theorem.

Theorem 3.1. In order for system (1.1) to belong to class M it is necessary that

$$\mathrm{Im}\,g \equiv \mathrm{Im}\left(\frac{\pi_{hj}^{(-n)}}{\omega_j} + i\nu_{hj}^{(-n)}\right)\left(\frac{\pi_{jh}^{(n)}}{\omega_h} - i\nu_{jh}^{(n)}\right) = 0 \tag{3.5}$$

Theorem 3.2. In order for system (1.1) to belong to class M for all possible positive definite diagonal matrices C, it is necessary that in (1.1)

$$\arg \pi_{hj}^{(n)} = \arg \pi_{jh}^{(n)} = \arg \nu_{hj}^{(n)} - \frac{\pi}{2} = \arg \nu_{jh}^{(n)} - \frac{\pi}{2} \pmod \pi \tag{3.6}$$

Conditions (3.5) and (3.6) are fulfilled in cases (1.3) and (1.4).

For a system (1.1) of class M without friction, on the boundary of the region of instability the expression under the radical sign is equal to zero. Hence, it follows that

$$\lambda = \frac{1}{2n}\left[\frac{\pi_{jj}^{(0)}}{\omega_j} + \frac{\pi_{hh}^{(0)}}{\omega_h} \pm 2\sqrt{g}\right] \tag{3.7}$$

where g is determined in (3.3). Let us consider the case of a system (1.1) of class M in which $N(\tau) \equiv 0$

$$\frac{d^2Y}{dt^2} + (C + \mu P(\theta t))\, Y = 0 \qquad (\mu \geqslant 0) \tag{3.8}$$

From formula (3.7), where $\nu_{jh}^{(-n)} = \nu_{jh}^{(n)} = 0$, follows the theorem.

Theorem 3.3. For a system of equations (3.8) of class M with a positive definite diagonal matrix C:

1) The frequency $\theta = \theta_{n,j,h}$ (2.1) will be strongly stable for

$$\pi_{hj}^{(-n)}\pi_{jh}^{(n)} < 0 \tag{3.9}$$

2) The frequency $\theta = \theta_{n,j,h}$ (2.1) will be strongly unstable for

$$\pi_{hj}^{(-n)}\pi_{jh}^{(n)} > 0 \tag{3.10}$$

3) The frequency $\theta = \theta^*_{n,j,h}$ (2.2) will be strongly stable if (3.10) is fulfilled.

4) The frequency $\theta = \theta^*_{n,j,h}$ (2.2) will be strongly unstable if (3.9) is fulfilled.

The last two statements of the theorem follow from the fact that for the conjugate frequence $\theta^*_{n,j,h}$, the expression for g (3.3) changes the sign.

System (3.8) with a positive definite matrix C can be reduced to the system with a diagonal matrix C by a linear transformation of form (1.5). Hence, from Theorem 3.3 follows the theorem.

Theorem 3.4. If for a system of equations (3.8) of class M with a positive definite matrix C one of the frequencies θ_0 and $\theta_0{}^*$ (2.1) and (2.2) is strongly stable, then its conjugate frequency will be strongly unstable; and *vice versa*, if one of the frequencies θ_0 and $\theta_0{}^*$ is strongly unstable, then its conjugate frequency will be strongly stable (provided that the assumptions made at the beginning of the section are fulfilled).

Example 3.1. For a system of two equations

$$\frac{d^2y_1}{dt^2} + \omega_1^2 y_1 + \mu(\cos\theta t - 2\cos 2\theta t)\, y_2 = 0 \tag{3.11}$$

$$\frac{d^2y_2}{dt^2} + \omega_2^2 y_2 + \mu(-3\cos\theta t + 4\cos 2\theta t)\, y_1 = 0, \qquad \omega_1 > \omega_2$$

by comparison with (1.1) and (1.2) we find

$$\pi_{12}^{(1)} = 0.5, \qquad \pi_{21}^{(1)} = -1.5, \qquad \pi_{12}^{(2)} = -1, \qquad \pi_{21}^{(2)} = 2 \tag{3.12}$$

From Theorem 3.3 it follows that the frequencies

$$\theta_1 = \omega_1 + \omega_2, \qquad \theta_2 = 0.5(\omega_1 + \omega_2) \tag{3.13}$$

are strongly stable, and the frequencies

$$\theta_1^* = \omega_1 - \omega_2, \qquad \theta_2^* = 0.5(\omega_1 - \omega_2) \tag{3.14}$$

are the only strongly unstable ones.

For a canonical system of differential equations (3.8), where $P(\tau) = P^*(\tau)$, we have

$$\pi_{hj}^{(n)} = \pi_{jh}^{(n)}, \qquad g = \pi_{hj}^{(-n)}\pi_{jh}^{(n)} = |\pi_{jh}^{(n)}|^2 \geqslant 0 \tag{3.15}$$

Therefore, the following theorem is valid.

Theorem 3.5. If in a system (3.8) of class M with a diagonal positive definite matrix C:

1) The matrix $P(\tau)$ is symmetric $P(\tau) = P^*(\tau)$, then the frequencies θ_0 (2.1) cannot be strongly stable, and their conjugate frequencies θ_0^* (2.2) cannot be strongly unstable.

2) The matrix $P(\tau)$ is skew-symmetric $P(\tau) = -P^*(\tau)$, then the frequencies θ_0 (2.1) cannot be strongly unstable, and the frequencies θ_0^* (2.2) cannot be strongly stable.

Statement 1 follows from a theorem due to Krein [2, p.353] and [4, p.493].

Note that if matrices $P_1(\tau)$ and $P_2(\tau)$ in (3.8) cause some frequency θ_0 to be strongly unstable, the total perturbation matrix $P(\tau)$

$$P(\tau) = P_1(\tau) + P_2(\tau) \tag{3.16}$$

may make the same frequency strongly stable.

We are assuming that all the systems of equations (3.8) with matrices $P_1(\tau)$, $P_2(\tau)$ and $P(\tau)$ (3.16) are of class M.

Example 3.2. For a system of two differential equations (3.8) the frequency $\theta_0 = \omega_1 + \omega_2$ will be strongly unstable for the perturbation matrices $P_2(\tau)$ and $P_1(\tau)$

$$P_1(\theta t) = \begin{pmatrix} 0 & 2\cos\theta t \\ \cos\theta t & 0 \end{pmatrix}, \qquad P_2(\theta t) = \begin{pmatrix} 0 & -\cos\theta t \\ -2\cos\theta t & 0 \end{pmatrix} \tag{3.17}$$

For a system (3.8) with the total perturbation matrix $P(\tau)$ (3.16) the frequency $\theta_0 = \omega_1 + \omega_2$ will be strongly stable.

Consider a system (1.1) of class M, if $P(\tau) \equiv 0$

$$d^2Y / dt^2 + \mu N(\theta t)\, dY / dt + CY = 0 \tag{3.18}$$

The following theorems are the consequence of formulas (3.7) and (3.3).

Theorem 3.6. For a system of equations (3.18) of class M with a diagonal positive definite matrix C:

1) The frequencies $\theta_{n,j,h}$ (2.1) and $\theta^*_{n,j,h}$ (2.2) will be strongly stable for

$$\nu_{hj}^{(-n)} \nu_{jh}^{(n)} < 0 \tag{3.19}$$

2) The frequencies $\theta_{n,j,h}$ (2.1) and $\theta^*_{n,j,h}$ (2.2) will be strongly unstable for

$$\nu_{hj}^{(-n)} \nu_{jh}^{(n)} > 0 \tag{3.20}$$

Theorem 3.7. For a system of equations (3.18) of class M with an arbitrary positive definite matrix C, both conjugate frequencies $\theta_{n,j,h}$ and $\theta^*_{n,j,h}$ are either simultaneously strongly stable, or simultaneously strongly unstable, or are simultaneously neither strongly stable, nor strongly unstable.

Theorem 3.8. Let the matrix C in a system (3.18) of class M be diagonal and positive definite.

1) *If the matrix $N(\tau)$ is symmetric $N^* = N$, then all the frequencies* $\theta_{n,j,h}$ (2.1) and $\theta^*_{n,j,h}$ (2.2) cannot be strongly stable.

2) If the matrix $N(\tau)$ is skew-symmetric $N^* = -N$, then all the frequencies $\theta_{n,j,h}$ (2.1) and $\theta^*_{n,j,h}$ (2.2) cannot be strongly unstable.

The proof of the theorem follows from the fact that for a symmetric matrix $N(\tau)$ condition (3.20) is fulfilled, and for a skew-symmetric matrix condition (3.19) is fulfilled, including the equality sign in the case $\nu_{jh}^{(n)}\nu_{hj}^{(-n)} = 0$.

Example 3.3. For a system of differential equations

$$\begin{aligned} d^2y_1 / dt^2 + \mu (2\sin\theta t - 4\sin 2\theta t)\, dy_2 / dt + \omega_1^2 y_1 = 0 \\ d^2y_2 / dt^2 + \mu (6\sin\theta t + 8\sin 2\theta t)\, dy_1 / dt + \omega_2^2 y_2 = 0 \end{aligned} \qquad (\omega_1 > \omega_2) \tag{3.21}$$

by comparison with (1.2) we find

$$\nu_{12}^{(1)} = -i, \qquad \nu_{12}^{(2)} = 2i, \qquad \nu_{21}^{(1)} = -3i, \qquad \nu_{21}^{(2)} = -4i \tag{3.22}$$

The frequencies

$$\theta_1 = \omega_1 + \omega_2, \qquad \theta_1^* = \omega_1 - \omega_2 \tag{3.23}$$

will be strongly stable, and the frequencies

$$\theta_2 = 0.5\,(\omega_1 + \omega_2), \qquad \theta_2^* = 0.5\,(\omega_1 - \omega_2) \tag{3.24}$$

will be strongly unstable.

A similar simple relation of stability for mutually conjugate frequencies does not exist in the general case of a system (1.1). Note that the perturbations, which are the matrices $N(\tau)$ and $P(\tau)$, which, acting separately, cause neither strong stability nor strong instability, may cause both effects when acting together. Obviously, the quantity g (3.3) is an invariant for transformations of the form (1.5).

4. Let us consider the stability of equations of class M with friction. The inequality containing real a, b and c

$$\operatorname{Im} \sqrt{a + ib} > c > 0 \tag{4.1}$$

can be transformed into the form

$$b^2 > 4c^2 (c^2 + a) \tag{4.2}$$

Solutions of a system of equations (1.1) may be unstable for $\nu_{jj}^{(0)} > 0$ and $\nu_{hh}^{(0)} > 0$, if

$$\operatorname{Im}\left[\left(\frac{\pi_{jj}^{(0)}}{\omega_j} + \frac{\pi_{hh}^{(0)}}{\omega_h} + i\,(\nu_{jj}^{(0)} - \nu_{hh}^{(0)}) - 2\lambda n\right)^2 - 4g\right]^{1/2} > (\nu_{jj}^{(0)} + \nu_{hh}^{(0)}) > 0 \tag{4.3}$$

Corresponding to inequality (4.3), an inequality of the form (4.2), solved for λ becomes

$$\lambda_- < \lambda < \lambda_+ \tag{4.4}$$

$$\lambda_\pm = \frac{1}{2n}\left[\frac{\pi_{jj}^{(0)}}{\omega_j} + \frac{\pi_{hh}^{(0)}}{\omega_h} \pm \left(\frac{(\nu_{jj}^{(0)} + \nu_{hh}^{(0)})^2}{\nu_{jj}^{(0)}\nu_{hh}^{(0)}} g - (\nu_{jj}^{(0)} + \nu_{hh}^{(0)})^2\right)^{1/2}\right] \tag{4.5}$$

where g is defined by formula (3.3). The value of the expression under the radical sign in (4.5) for arbitrarily small values of $\nu_{jj}^{(0)} > 0$, $\nu_{hh}^{(0)} > 0$ may be arbitrarily large. The expansion of the region of instability may take place only for $\nu_{jj}^{(0)} \neq \nu_{hh}^{(0)}$ and $j \neq h$, i.e. in the case of combination resonance. The expansion itself occurs on account of a change in the relation between $\nu_{jj}^{(0)}$ and $\nu_{hh}^{(0)}$.

This interesting phenomenon was first noticed in [1] where its basic properties were demonstrated on a system of equations much simpler than those of class M.

If $g < 0$, i.e. if the solutions of system (1.1) without friction are strongly stable, they will remain strongly stable (4.5) when friction is introduced. Let $g > 0$, i.e. let the frequency θ_0 be strongly unstable. From (3.4) we find the expansion of quantities Im $z_{1,2}$ for small $\nu_{jj}^{(0)}$ and $\nu_{hh}^{(0)}$

$$\operatorname{Im} z_{1,2} = {}^1/_4\,[\nu_{jj}^{(0)} + \nu_{hh}^{(0)} \pm \alpha\,(\alpha^2 - 4g)^{-1/2}(\nu_{jj}^{(0)} - \nu_{hh}^{(0)})] + O\,(\,|\,\nu_{jj}^{(0)}\,|^2 + |\,\nu_{hh}^{(0)}\,|^2) \tag{4.6}$$

where

$$\alpha = \frac{\pi_{jj}^{(0)}}{\omega_j} + \frac{\pi_{hh}^{(0)}}{\omega_h} - 2\,\lambda n \tag{4.7}$$

In the region of stability of system (1.1) without friction, i.e. for $\alpha^2 - 4g > 0$ when $g > 0$ one can always choose the numbers $\nu_{jj}^{(0)}$, $\nu_{hh}^{(0)}$ in such a way that the coefficient at the imaginary part of one of the roots $z_{1,2}$ will be negative. Then the solutions will become unstable.

Comparing formulas (1.6) and (3.7) we arrive at the following theorem.

Theorem 4.1. If in a system (1.1) of class M with a positive definite diagonal matrix C a certain frequency $\theta_0 = \theta_{n,j,h}$ (2.1) has a wide range of instability adjacent to it

$$\theta_0 + \mu\lambda_1 + O\,(\mu^2) < \theta < \theta_0 + \mu\lambda_1 + O\,(\mu^2), \quad \lambda_1, \lambda_2 = \text{const}, \quad \lambda_1 - \lambda_2 \neq 0 \tag{4.8}$$

then upon introduction of friction $\nu_{jj}^{(0)} > 0$ and $\nu_{hh}^{(0)} > 0$, the boundaries $\theta_{1,2}$ of the region of instability in the plane of parameters μ, θ

takes the form

$$\theta_{1,2} = \theta_0 + \mu\left(\frac{\lambda_1+\lambda_2}{2}\right) \pm \mu\,(\nu_{jj}^{(0)} + \nu_{hh}^{(0)})\left(\frac{(\lambda_2-\lambda_1)^2}{4\nu_{jj}^{(0)}\cdot\nu_{hh}^{(0)}} - \frac{1}{4n^2}\right)^{1/2} + O(\mu^2) \qquad (4.9)$$

Theorem 4.2. If in a system of equations (1.1) of class M with a positive definite matrix C:

1) A certain frequency θ_0 is strongly stable, then the introduction of sufficiently small friction will keep it strongly stable.

2) A certain combination frequency is strongly unstable, then the introduction of friction can always lead to a widening of the region of instability for sufficiently small values of $\mu > 0$.

Theorem 4.2 shows the special danger of combination resonances, since in real systems a small friction is always present.

In the relatively recent past the combination frequency was not being given sufficient consideration, as for instance in [7].

5. Let us look into the analyticity of the boundaries of the instability region. Suppose that the boundaries of the region of instability in the plane μ, θ for a system (1.1) of class M are found by the method of small parameter. The question arises about the possibility of constructing the boundaries of the region of instability $\theta_1(\mu)$ and $\theta_2(\mu)$, where $\theta_1(0) = \theta_2(0) = \theta^0$ in the form of series of integer powers of μ. If in equation (3.2) the infinitesimal terms of higher order are retained, then in formula (3.7) the expression under the radical sign will contain an additional function $O(\mu)$. For the equations of class M this function will be real for $\operatorname{Im} z = \operatorname{Im}\lambda = 0$. If $g > 0$, then, setting the expression under the radical sign in (3.7) equal to zero, one can always solve the resulting equation for λ. We obtain two real expressions for λ, i.e. for $(\theta_{1,2}(\mu) - \mu^0)\mu^{-1}$ which are analytic for $\mu = 0$. Hence, we have the theorem.

Theorem 5.1. If in a system of equations (1.1) of class M for a certain resonant frequency θ_0 or $\theta_0{}^*$ (2.1) and (2.2) (assuming that the conditions of Section 3 are fulfilled) the expression for g (3.3) is positive, then this strongly unstable frequency will border on a region of instability with boundaries $\theta_1(\mu)$ and $\theta_2(\mu)$. Here $\theta_1(\mu)$ and $\theta_2(\mu)$ are analytic functions of μ in the sufficiently small neighborhood of the point $\mu = 0$.

Note 5.1. The condition $g > 0$ is in a way a necessary one, as can be seen from the following example. With additional restrictions imposed upon a system (1.1) of class M (for instance, the requirement that the equations be canonical) the condition $g > 0$ will only be sufficient for the analyticity of $\theta_1(\mu)$ and $\theta_2(\mu)$ at the point $\mu = 0$.

Example 5.1. Consider a system of differential equations

$$d^2y_1/dt^2 + \omega_1^2 y_1 + 2\alpha_1 y_2 \cos\theta t = 0, \qquad d^2y_2/dt^2 + \omega_2^2 y_2 + 2\alpha_2 y_1 \cos\theta t = 0 \tag{5.1}$$
$$\omega_1 > \omega_2, \qquad \theta_0 = \omega_1 + \omega_2, \qquad \alpha_1 \approx 0, \qquad \alpha_2 \approx 0$$

Expressions (2.12) with accuracy up to $O(\alpha_1^2\alpha_2^2)$ have the form

$$a_{11} = p^2 + \omega_1^2 - \alpha_1\alpha_2 [(p + \theta i)^2 + \omega_2^2]^{-1} + O(\alpha_1^2\alpha_2^2), \qquad \alpha_{12} = \alpha_1 \tag{5.2}$$
$$a_{22} = (p - \theta i)^2 + \omega_2^2 - \alpha_1\alpha_2 [(p - 2\theta i)^2 + \omega_2^2)]^{-1} + O(\alpha_1^2\alpha_2^2), \qquad \alpha_{21} = \alpha_2$$

Setting $p = i\omega_1 + iz$ and $\theta = \theta_0 + \lambda$, we obtain from (2.13)

$$\lambda_{1,2} = \pm \sqrt{\frac{\alpha_1\alpha_2}{\omega_1\omega_2}} + \frac{\alpha_1\alpha_2}{4\omega_1\omega_2(\omega_1 + \omega_2)} + O(\alpha^{3/2}\alpha^{3/2}) \tag{5.3}$$

If $\alpha_1 = \mu a_1$, $\alpha_2 = \mu^2 a_2$, $a_1 a_2 > 0$, then the equations of the boundaries (5.3) for $\mu > 0$ take the form

$$\lambda = \pm \frac{\mu^{3/2}\sqrt{a_1a_2}}{\sqrt{\omega_1\omega_2}} + \frac{\mu^3 a_1a_2}{4\omega_1\omega_2(\omega_1 + \omega_2)} + O(\mu^{9/2}) \tag{5.4}$$

In this case $g = 0$, the expressions $\theta_1(\mu)$ and $\theta_2(\mu)$ for the boundaries have an algebraic singular point for $\mu = 0$.

In conclusion, let us note that some introductory questions were treated in a fashion analogous to [8], and the analyticity of the boundary for a canonical system was investigated in [9].

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